Rational Functions and Relations

Overview

Number of instruction days: 12–14  (1 day = 53 minutes)

Content to Be Learned

- Rewrite simple rational expressions in different forms.
- Add, subtract, multiply and divide rational expressions.
- Create simple rational equations and inequalities.
- Solve simple rational equations in one variable.
- Graph rational functions.
- Represent and solve pairs of polynomial, rational, radical, absolute-value, and exponential equations graphically.
- Use the conceptual understanding of functions to find the inverse, of a simple rational function, if the inverse exists.

Mathematical Practices to Be Integrated

5 Use appropriate tools strategically.
- Use a graphing calculator to graph rational functions.
- Use a graphing calculator to represent and solve systems of equations graphically.

7 Look for and make use of structure.
- Recognize and use the similarities between properties of rational numbers and rational algebraic expressions to develop techniques for performing operations with rational expressions.

Essential Questions

- What are the similarities and differences between simplifying rational expressions and simplifying rational numbers?
- How can rational equations be applied in real-world situations?
- Where would you find the solutions to systems of polynomial, rational, radical, absolute-value, and exponential equations represented?
the graph of a rational expression? 

- What is the reflective line of symmetry for an inverse function?

### Standards

**Common Core State Standards for Mathematical Content**

**Algebra**

**Creating Equations**

Create equations that describe numbers or relationships [Equations using all available types of expressions, including simple root functions]

A-CED.1 Create equations and inequalities in one variable and use them to solve problems. Include equations arising from linear and quadratic functions, and simple rational and exponential functions.*

**Arithmetic with Polynomials and Rational Expressions**

Rewrite rational expressions [Linear and quadratic denominators]

A-APR.6 Rewrite simple rational expressions in different forms; write \( \frac{a(x)}{b(x)} \) in the form \( g(x) + \frac{r(x)}{b(x)} \), where \( a(x), b(x), q(x) \), and \( r(x) \) are polynomials with the degree of \( r(x) \) less than the degree of \( b(x) \), using inspection, long division, or, for the more complicated examples, a computer algebra system.

**Reasoning with Equations and Inequalities**

Understand solving equations as a process of reasoning and explain the reasoning [Simple radical and rational]

A-REI.2 Solve simple rational and radical equations in one variable, and give examples showing how extraneous solutions may arise.

Represent and solve equations and inequalities graphically [Combine polynomial, rational, radical, absolute value, and exponential functions]
A-REI.11 Explain why the $x$-coordinates of the points where the graphs of the equations $y = f(x)$ and $y = g(x)$ intersect are the solutions of the equation $f(x) = g(x)$; find the solutions approximately, e.g., using technology to graph the functions, make tables of values, or find successive approximations. Include cases where $f(x)$ and/or $g(x)$ are linear, polynomial, rational, absolute value, exponential, and logarithmic functions.*

Functions

Building Functions F-BF

Build new functions from existing functions [Include simple radical, rational, and exponential functions; emphasize common effect of each transformation across function types]

F-BF.4 Find inverse functions.

a. Solve an equation of the form $f(x) = c$ for a simple function $f$ that has an inverse and write an expression for the inverse. For example, $f(x) = 2x^3$ or $f(x) = (x+1)/(x-1)$ for $x \neq 1$.

Interpreting Functions F-IF

Interpret functions that arise in applications in terms of the context [Using technology and appropriate models]

F-IF.4 For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship. Key features include: intercepts; intervals where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior; and periodicity.*

Common Core State Standards for Mathematical Practice

5 Use appropriate tools strategically.

Mathematically proficient students consider the available tools when solving a mathematical problem. These tools might include pencil and paper, concrete models, a ruler, a protractor, a calculator, a spreadsheet, a computer algebra system, a statistical package, or dynamic geometry software. Proficient students are sufficiently familiar with tools appropriate for their grade or course to make sound decisions about when each of these tools might be helpful, recognizing both the insight to be gained and their limitations. For example, mathematically proficient high school students analyze graphs of functions and solutions generated using a graphing calculator. They detect possible errors by strategically using estimation and other mathematical knowledge. When making mathematical models, they know that technology can enable them to visualize the results of varying assumptions, explore consequences, and compare predictions with data. Mathematically proficient students at various grade levels are able to identify relevant external mathematical resources, such as digital content located on a
website, and use them to pose or solve problems. They are able to use technological tools to explore and deepen their understanding of concepts.

7 Look for and make use of structure.

Mathematically proficient students look closely to discern a pattern or structure. Young students, for example, might notice that three and seven more is the same amount as seven and three more, or they may sort a collection of shapes according to how many sides the shapes have. Later, students will see 7 \times 8 equals the well remembered 7 \times 5 + 7 \times 3, in preparation for learning about the distributive property. In the expression \( x^2 + 9x + 14 \), older students can see the 14 as 2 \times 7 and the 9 as 2 + 7. They recognize the significance of an existing line in a geometric figure and can use the strategy of drawing an auxiliary line for solving problems. They also can step back for an overview and shift perspective. They can see complicated things, such as some algebraic expressions, as single objects or as being composed of several objects. For example, they can see \( 5 - 3(x - y)^2 \) as 5 minus a positive number times a square and use that to realize that its value cannot be more than 5 for any real numbers \( x \) and \( y \).

Clarifying the Standards

Prior Learning

In Grade 3, students began to study rational numbers and fractions, and they represented fractions \( 1/b \) and \( a/b \) on a number line. They also examined equivalent and unequal fractions. In Grade 4, students extended their understanding of unit fractions to build fractions less than 1, added and subtracted fractions with like denominators, and multiplied fractions by whole numbers. In Grade 5, students used equivalent fractions as a strategy to add and subtract fractions with unlike denominators, and they multiplied and divided fractions using their previous understanding of multiplication and division. In Grade 6, students divided fractions by fractions and found common factors and multiples. Additionally, they solved one-variable equations and inequalities for the first time. In Grade 7, students learned to solve multistep equations and inequalities in word problems. In Grade 8, students solved linear equations in a variety of forms including those with rational coefficients. They also solved systems of linear equations graphically and algebraically. In Algebra I, students learned the general principle of solving equations while mastering linear equations and inequalities, and they studied a variety of function types, including linear, quadratic, exponential, and absolute value. Students solved systems of linear and quadratic equations graphically. Finally, students developed the structural similarities between integers and polynomials.

Current Learning

Modeling with functions is a critical area for Algebra II. Students identify appropriate types of functions for given data, adjust parameters as needed, and compare the quality of models. In Algebra II, students learn how to use operations and solve rational equations. In this unit, students develop skills in graphing and solving rational equations. They are expected to evaluate and simplify rational expressions, solve
rational equations, and find inverse functions, if they are viable. Finally, students solve systems of equations using a variety of function types (polynomial, rational, radical, absolute value, and exponential) graphically. Understanding solving equations as a process of reasoning and explaining the reasoning, and representing and solving equations and inequalities graphically are classified as major content in the PARCC Model Frameworks for Mathematics. Building new functions from existing functions is defined as additional content in the PARCC frameworks. Creating equations that describe numbers or relationships and rewriting rational expressions is defined as supporting content.

Future Learning

In Precalculus, students will learn to graph and interpret key features of the graphs of rational functions, including zeros, asymptotes, and end behavior. In AP Calculus BC, students will use partial fraction decomposition to write a rational expression as the sum of simpler rational expressions; the result can then be used for evaluating integral expressions or for determining the sum of an infinite telescoping series. Knowledge of rational expressions will be helpful in careers involving chemistry, physics, biochemistry, and molecular studies where rational expressions will be used for analyzing data. Rational expressions are also found in topics such as optics, aerodynamics, and audio systems.

Additional Findings

There are no additional findings for this unit.

Assessment

When constructing an end-of-unit assessment, be aware that the assessment should measure your students' understanding of the big ideas indicated within the standards. The CCSS for Mathematical Content and the CCSS for Mathematical Practice should be considered when designing assessments. Standards-based mathematics assessment items should vary in difficulty, content, and type. The assessment should comprise a mix of items, which could include multiple choice items, short and extended response items, and performance-based tasks. When creating your assessment, you should be mindful when an item could be differentiated to address the needs of students in your class.

The mathematical concepts below are not a prioritized list of assessment items, and your assessment is not limited to these concepts. However, care should be given to assess the skills the students have developed within this unit. The assessment should provide you with credible evidence as to your students' attainment of the mathematics within the unit.

- Rewrite rational expressions in equivalent forms.
- Simplify rational expressions using addition, subtraction, multiplication and division.
- Create and solve simple rational equations and inequalities and use them in a contextual situation to solve problems.
• Identify and analyze key features of graphs of simple rational functions and transformations with and without technology, including intercepts, domain/range, asymptotes, and points of discontinuity; symmetries; and end behavior.

• Identify and write equations for inverses of simple rational functions.

• Represent and find the solution of pairs of equations by graphing. Include combinations of the following functions: polynomial, rational, radical, absolute value, and exponential.

**Instruction**

**Learning Objectives**

Students will be able to:

• Use the arithmetic operations multiplication and division to combine rational expressions.

• Use the arithmetic operations addition and subtraction to combine rational expressions.

• Analyze graphs of simple rational functions and transformations with and without technology, including intercepts, domain/range, asymptotes, and points of discontinuity; symmetries; and end behavior.

• Create and solve simple rational equations and inequalities.

• Find the inverse of a rational function if an inverse function exists.

• Reflect on and demonstrate understanding of rational expressions and functions.

**Resources**


• Sections 9-1 through 9-4 (pp. 553 - 584)

• Section 9-6 (pp. 594 – 602)


• Sections 9-4: Lab: Graphing Rational Functions: TI-Nspire

• Section 9-6: Lab: Solving Rational Equations and Inequalities: TI-Nspire Chapter 9 Resource Masters, (pp. 5 - 31, 39 - 44)
Illustrative Mathematics: The following tasks may be used for finding inverses of rational functions. See the Supplementary Materials Section of this binder for the PDF versions of these tasks.

- Invertible or Not: http://www.illustrativemathematics.org/illustrations/1374
- Temperature Conversions: http://www.illustrativemathematics.org/illustrations/364
- Temperatures in Degrees Fahrenheit and Celsius: http://www.illustrativemathematics.org/illustrations/501

Exam View Assessment Suite

TI-Nspire Teacher Software

www.khanacademy.org

Education.TI.com: Exploring Rational Functions: ID 8968. See the Supplementary Materials Section of this binder for the Student and Teacher materials.

Note: The district resources may contain content that goes beyond the standards addressed in this unit. See the Planning for Effective Instructional Design and Delivery section below for specific recommendations.

Materials

TI – Nspire graphing calculators

Instructional Considerations

Key Vocabulary

horizontal asymptote  points of discontinuity
rational equation  rational inequality
vertical asymptote

Planning for Effective Instructional Design and Delivery

Reinforced vocabulary taught in previous grades or units: inverse function, linear, quadratic, factoring, extraneous solution, and least common denominator.
As students begin this unit, be sure to make connections with using the multiplicative identity to rewrite rational expressions. Reiterating the connections to arithmetic operations used on all rational numbers. This should help students realize that rational expressions are just fractions with variables. Although some simple rational equations may be solved by using the proportion method of cross-multiplying ratios, many students develop misconceptions about solving rational equations. If there is another term added or subtracted to one side of the equation, students might disregard the addend and only cross-multiply the terms directly across the equal sign.

In this unit, students find the inverse of rational functions. Supplemental materials must be identified for the standard in which students must find inverse functions by solving an equation of the form, \( f(x) = c \) for a simple rational function that has an inverse, and write an expression for the inverse. An example provided in the standard is \( f(x) = \frac{x + 1}{x - 1} \) for \( x \neq 1 \). Similar problems can be generated using items found using the textbook and ancillary materials.

The Illustrative Mathematics Project was developed under the guidance of members of the Common Core State Standards working group as well as other national experts in mathematics and mathematics education (PARCC Model Content Frameworks, p. 11). The project provides illustrations and examples of tasks supporting the implementation of the Common Core State Standards. The tasks listed below can be used to support finding the inverse of rational functions and are available on the following website: http://illustrativemathematics.org/standards/hs. Direct links to each illustration is provided in the resources; lesson materials for the illustrations have also been included in the supplementary resource section of this curriculum framework binder.

- US Households
- Invertible or Not
- Temperature Conversions
- Temperatures in Degrees Fahrenheit and Celsius

Students should apply rational equations and inequalities in problem situations. Have students work together to solve “work” problems such as in section 9-6 in the Real World Example 5 on page 598 in the textbook. Consider using additional examples similar to the following: Latisha and Carlos mow lawns together to make money over the summer. Latisha working alone could complete a particular job in 4.5 hours, and Carlos could complete it alone in 3.7 hours. Have students find solutions to this problem by using both methods: proportional reasoning and rational equations. Students should identify similarities and differences by comparing the two solution methods. The following questions might be asked to elicit a deeper understanding of rational functions: How long will it take them to complete the job when they work together? What strategies are used to solve this rational problem using proportional reasoning? How can you determine what the excluded (extraneous) values are when solving a rational equation? Why is it necessary to check your results?
Note: Solving rational equations using proportional reasoning usually transforms the equation into another type of equation (linear, quadratic, etc.). Depending on the type of equation that results, there are different methods of solving. Integrating a nonlinguistic representation such as a graphic organizer can help organize the process for students. Students can be placed into cooperative groups of two to four and asked to create a graphic organizer for Steps to Solving a Rational Equation, along with an example illustrating each step (or students could be supplied with a teacher-made graphic organizer). Students can then share their organizers with the rest of the class. Instruct students to include all possible steps, provide an example illustrating each step, and a check verifying results and/or extraneous solutions. Examples of graphic organizers to help guide student work may be found on the internet.

The Differentiated Instruction section on page 561 in the Teacher’s Edition includes a suggestion for building the strong base required for the work with rational expressions. The questions are designed to make the connection of student’s prior work with arithmetic operations to their future work with rational expressions.

By comparing and contrasting the characteristics of rational functions under transformations with regard to effect on domain/range, direction (and stretch/compression, to some degree) of the hyperbolic branches, and shifting, students quickly begin to be able to predict their effects. Creating posters of examples of parent functions and functions using a particular transformation allows students to visually make these connections.

Using technology to solve equations and inequalities is helpful for all learners. The TI–Nspire activity, Exploring Rational Functions: ID 8968 is located in the Supplementary Unit Materials Section of this Binder. In this activity students investigate the graphs of functions of the form $y = \frac{1}{x-a}$. They discover that the graph of such a function has a vertical asymptote at $x = a$, and a horizontal asymptote at $y = 0$. Students also investigate the graphic and numeric consequences of such asymptotic behavior by observing a trace point on the graph move in response to various inputs for the independent variable.

Additional TI-Nspire resources can be found using the TI-Nspire Teacher Software. The content tab of the TI-Nspire desktop software contains links to Algebra 2, Glencoe McGraw-Hill, 2010 textbook. These resources are accessible by chapter and section.

Use nonlinguistic representations such as word walls, foldables, or graphic organizers to support the learning of vocabulary. Vocabulary support is essential to student success in Algebra 2. A foldable study organizer, such as the one illustrated on page 744, can also be used to assist students with note taking skills. Students can create and use a foldable to take notes, define terms, capture key concepts and ideas, and write illustrated examples. In this unit, there are numerous opportunities to differentiate instruction and scaffold support using the Study Notebook.

The Glencoe Online Personal Tutors, can also be used to supplement classroom instruction and are located on the website http://connected.mcgraw-hill.com/connected/.